

TOWARDS A “PRE-CANONICAL” QUANTIZATION OF GRAVITY
WITHOUT THE SPACE+TIME DECOMPOSITION

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Abstract. Quantization of gravity is discussed in the context of field quantization based on an analogue of canonical formalism (the De Donder-Weyl canonical theory) which does not require the space+time decomposition. Using Hořava’s (1991) De Donder-Weyl formulation of General Relativity we arrive to a covariant generalization of the Schrödinger equation for the wave *function* of space-time and metric variables and a supplementary “bootstrap condition” which self-consistently incorporates the classical background space-time geometry as a quantum average and closes the system of equations. Some open questions for further research are outlined.

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1 Introduction

Contemporary approaches to the quantum theory of gravity (for a recent review and references see, e.g., [1]) either start from the classical theory trying to quantize it by means of various available techniques, or seek to build a more general framework of which the quantum theory of gravity would be a byproduct or a limiting case. The attempts of the first kind include quantum geometrodynamics based on the Wheeler-De Witt equation, Ashtekar’s program of nonperturbative canonical quantum gravity and approaches based on the path integral. String/M- theory and, in a sense, models inspired by non-commutative geometry represent approaches of the second kind. The

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goal of all this is a synthesis of quantum theory and general relativity either in the sense of a fusion into a “quantum general relativity” or incorporation into a more general unifying theory of all interactions.

The existing attempts to *quantize* gravity are known to be confronted by both the problematic mathematical meaning of the involved constructions and the conceptual questions originating in difficulties of reconciling the fundamental principles of quantum theory with those of general relativity (see e.g. [2] for a review and further references).¹ In particular, a distinct role of the time dimension in the probabilistic interpretation of quantum theory and in the formulation of quantum evolution laws seems to contradict the equal rights status of space-time dimensions in the theory of relativity. Moreover, *canonical quantization* is preceded by the Hamiltonian formulation which requires the singling out of a time parameter or, more technically, the global hyperbolicity of space-time. The latter assumption, however, looks rather unnatural for the quantum fluctuating “space-time foam” of quantum gravity, that may indicate that the applicability to quantum gravity of procedures based on, or inspired by, the standard Hamiltonian framework can be rather limited.

These difficulties could be overcome if one would have in our disposal a quantization procedure in field theory which does not so sensibly depend on the singling out of a time parameter. However, such a “timeless” procedure, if understood as a generalized version of canonical quantization, clearly requires a version of canonical formalism without the distinct role of the time dimension and thus independent of the picture of fields as infinite dimensional systems evolving in time from the initial Cauchy data given on a space-like hypersurface.

Fortunately, although this seems to be not yet commonly known in theoretical physics, the Hamiltonian-like reformulations of the field equations which could be appropriate for a “timeless” version of canonical formalism have been known in the calculus of variations already at least since the thirties. In the simplest of them, the so-called De Donder-Weyl (DW) formulation [4, 5], the Euler-Lagrange field equations are written in the form

$$\partial_\mu y^a = \frac{\partial H}{\partial p_a^\mu}, \quad \partial_\mu p_a^\mu = -\frac{\partial H}{\partial y^a}, \quad (1.1)$$

where y^a denote field variables, $p_a^\mu := \partial L / \partial (\partial_\mu y^a)$ are to be referred to as *polymomenta*, $H := \partial_\mu y^a p_a^\mu - L$ is a function of (y^a, p_a^μ, x^ν) to be called the *DW Hamiltonian function*, and $L = L(y^a, \partial_\mu y^a, x^\nu)$ is a Lagrangian density. In this formulation the analogue of the configuration space is a finite dimensional space of space-time and field variables (x^ν, y^a) and the analogue of the extended phase space is a finite dimensional space of variables (y^a, p_a^μ, x^ν) called the (extended) *polymomentum phase space*. Note, that in formulation (1.1) fields are essentially described as a sort of multi-parameter generalized Hamiltonian systems rather than as infinite dimensional mechanical systems, as in the standard Hamiltonian formalism. In doing so the DW Hamiltonian function H , which thus far does not appear to have any evident physical interpretation, in a sense controls

¹Note, however, that the quantum field theory of gravity can be formulated in the low energy domain as an effective field theory [3].

the space-time *variations* of fields, as specified by equations (1.1), rather than their time *evolution*. However, the latter is implicit in (1.1) in the case of hyperbolic field equations for which the Cauchy problem can be posed.

It is interesting to note that for the set of DW canonical equations (1.1) there exists an analogue of the Hamilton-Jacobi theory. The corresponding DW Hamilton-Jacobi equation [4, 6] is formulated for n (n =the number of space-time dimensions) functions $S^\mu = S^\mu(y^a, x^\mu)$:

$$\partial_\mu S^\mu + H(x^\mu, y^a, p_a^\mu = \partial S^\mu / \partial y^a) = 0, \quad (1.2)$$

and naturally leads to the question as to which formulation of quantum field theory could yield this field theoretic Hamilton-Jacobi equation in the classical limit.

It should be mentioned that the DW formulation is a particular case of more general *Lepagean* canonical theories for fields which differ by the definitions of polymomenta and analogues of Hamilton's canonical function (both essentially follow from different choices of the Lepagean equivalents of the Poincaré-Cartan form; for further details and references see [7–9] and reviews quoted in [4]). All theories of this type treat space and time variables on equal footing and are finite dimensional in the sense that the corresponding analogues of the configuration and the phase space are finite dimensional. In addition, they all reduce to the Hamiltonian formalism of mechanics when $n = 1$. In a sense, all these formulations are intermediate between the Lagrangian formulation and the canonical Hamiltonian one. They still keep space-time variables indistinguishable but already possess essential features of the Hamiltonian-like description such as a reference to the first order form of the field equations and a Legendre transform. Moreover, there seem to be intimate relations, not fully studied as yet, between structures of the canonical Hamiltonian formalism and those of the Lepagean formulations. For this reason, let us refer to these formulations and the related structures as “*pre-canonical*”. Further justification of the term will be given in Conclusion.

It is worthy of noticing that pre-canonical formulations typically have different regularity conditions than the standard Hamiltonian formalism. For example, the DW formulation (1.1) implies that $\det ||\partial^2 L / \partial_\mu y^a \partial_\nu y^b|| \neq 0$ which is different from the usual requirement that $\det ||\partial^2 L / \partial_t y^a \partial_t y^b|| \neq 0$. As a result, the constraints understood as obstacles to the corresponding generalized Legendre transforms $\partial_\mu y^a \rightarrow p_a^\mu$ have a quite different structure so that singular theories from the point of view of the conventional formalism can be regular from the pre-canonical point of view (as, e.g., the Nambu-Goto string [16]) or vice versa (as, e.g., the Dirac spinor field [6]). This opens an yet unexplored possibility of avoiding the constraints analysis when quantizing within the pre-canonical framework by choosing for a given theory an appropriate non-singular Lepagean Legendre transform (in fact, this possibility is exploited below when quantizing General Relativity without any mention of constraints).

The idea of quantization proceeding from the DW canonical theory for fields dates back to Born and Weyl [25] but has not received much attention since then (see, however, [26]). Obviously, quantization needs more than just an existence of a Hamiltonian-like formulation of the field equations. Additional structures, such as the Poisson bracket (for canonical or deformation quantization), the symplectic structure (for geo-

metric quantization), and a Poisson bracket formulation of the field equations (in order to formulate the quantum dynamics law) are necessary. In spite of a number of earlier attempts [10] and related developments in the geometry of classical field theory based on the (Hamilton-)Poincaré-Cartan, or multisymplectic, form [11–13] (see [13] for more extensive references) and on another generalization of the symplectic structure due to Günther [14], a construction which could be suitable as a starting point of quantization was found only recently [15–17]. It leads to graded Poisson brackets defined on differential forms which represent in this case dynamical variables. From a more mathematical perspective, various relevant jet-bundle-theoretic constructions and generalizations of the symplectic geometry have been studied recently by a number of authors [18–24] who have significantly improved our knowledge of the underlying geometric structures of classical field theories, yet to be employed by physicists.

The elements of quantization in field theory based on the aforementioned brackets on differential forms have been discussed in [15, 28–30] and will be briefly summarized below. It should be noted that so far this is rather a preliminary approach some fundamental aspects of which, including a proper interpretation and a connection with the standard formalism of quantum field theory, are yet to be clarified. However, the approach possesses a distinct esthetic attraction, intriguing features, and as yet unexplored potential which make it worthy of further endeavors. The intrinsic finite dimensionality and manifest covariance could make the pre-canonical framework, if successful, a suitable complement to the presently available concepts and techniques of quantum field theory.

The main purpose of the present paper is to discuss a preliminary application of pre-canonical framework to the problem of quantization of General Relativity. We first, in Section 2, summarize basic aspects of pre-canonical formalism and quantization based on the DW theory and then, in Section 3, apply it to General Relativity. Discussion and concluding remarks are presented in Section 4.

2 Pre-canonical formalism and quantization based on DW theory

In this section we briefly summarize basic elements of the analogue of canonical formalism based on DW theory and then outline elements of quantization based on this formalism.

2.1 Classical theory

The mathematical structures underlying the DW form of the field equations, which can be suitable for quantization, have been studied in our previous papers [16, 17] to which we refer for more details.

The analogue of the Poisson bracket for the DW formulation can be deduced from the object called the *polysymplectic form* which in local coordinates can be written in

the form²

$$\Omega = -dy^a \wedge dp_a^\mu \wedge \omega_\mu$$

and is viewed as a field theoretic generalization of the symplectic form within the DW formulation. Note that if Σ , $\Sigma: (y = y_{in}(\mathbf{x}), t = t_{in})$, denotes the Cauchy data surface in the covariant configuration space the standard symplectic form in field theory, ω_S , can be expressed as the integral of the pull-back of Ω to Σ , $\Omega|_\Sigma$, i.e. $\omega_S = \int_\Sigma \Omega|_\Sigma$. The form Ω allows us to set up relations between p -forms $\overset{p}{F}$ and $(n-p)$ -multivectors (or more general algebraic operators of degree $-(n-p)$ on the exterior algebra) $\overset{n-p}{X}$:

$$\overset{n-p}{X} \lrcorner \Omega = d\overset{p}{F}. \quad (2.1)$$

Then the graded Poisson brackets of horizontal forms $\overset{p}{F} := \frac{1}{p!} F_{\mu_1 \dots \mu_p}(z^M) dx^{\mu_1} \dots \mu_p$, ($p = 0, 1, \dots, n$), can be defined by

$$\{\overset{p}{F}_1, \overset{q}{F}_2\} := (-)^{n-p} \overset{n-p}{X}_1 \lrcorner d\overset{q}{F}_2. \quad (2.2)$$

Hence the bracket of a p -form with a q -form is a form of degree $(p+q-n+1)$, where n is the dimension of the space-time. Consequently, the subspace of forms of degree $(n-1)$ is closed with respect to the bracket, as well as the subspace of forms of degree 0 and $(n-1)$.

The construction leads to a hierarchy of algebraic structures which are graded generalizations of the Poisson algebra in mechanics [16, 17]. Specifically, on a small subspace of horizontal forms called Hamiltonian forms (i.e. those which can be associated by relation (2.1) with multivectors) we obtain the structure of a Gerstenhaber algebra. Recall, that the latter is a triple $\mathcal{G} = (\mathcal{A}, \{\cdot, \cdot\}, \cdot)$, where \mathcal{A} is a graded commutative associative algebra with the product operation \cdot , $\{\cdot, \cdot\}$ is a graded Lie bracket fulfilling the graded Leibniz rule with respect to the product \cdot , with the degree of an element a of \mathcal{A} with respect to the bracket operation, $bdeg(a)$, and the degree of a with respect to the product \cdot , $pdeg(a)$, related as $bdeg(a) = pdeg(a) + 1$. In our case the bracket operation is a graded Lie bracket closely related to the Schouten-Nijenhuis bracket of multivector fields (the latter is related to our bracket in the similar way as the Lie bracket of vector fields is related to the Poisson bracket). Correspondingly, the graded commutative multiplication \bullet (the “co-exterior product”), with respect to which the space of Hamiltonian forms is stable, is given by

$$F \bullet G := *^{-1}(*F \wedge *G),$$

where $*$ is the Hodge duality operator. On more general (“non-Hamiltonian”) forms a non-commutative (in the sense of Loday’s “Leibniz algebras” [27]) higher-order (in

²Note, that this object can be understood as the equivalence class modulo forms of the highest horizontal degree n , see [16] for more details. Henceforth we denote $\omega := dx^1 \wedge \dots \wedge dx^n$, $\omega_\mu := \partial_\mu \lrcorner \omega = (-1)^{\mu-1} dx^1 \wedge \dots \widehat{dx^\mu} \dots \wedge dx^n$, $dx^{\mu_1} \dots \mu_p := dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$, and $\{z^M\} := \{y^a, p_a^\mu, x^\mu\}$.

the sense of a higher-order analogue of the graded Leibniz rule replacing the standard Leibniz rule in the definition) generalization of a Gerstenhaber algebra appears [17].

The bracket defined in (2.2) enables us to identify pairs of canonically conjugate variables and to represent the DW canonical equations in (generalized) Poisson bracket formulation. Namely, the appropriate notion of canonically conjugate variables in the present context is suggested by considering brackets of horizontal forms of the kind $y^a dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$ and $p_a^\mu \partial_\mu \lrcorner \partial_{\mu_1} \lrcorner \dots \lrcorner \partial_{\mu_q} \lrcorner \omega$ with $(p - q) \geq 0$. In particular, in the Lie subalgebra of Hamiltonian forms of degree 0 and $(n - 1)$ the non-vanishing “pre-canonical” brackets take the form [16]

$$\begin{aligned} \{p_a^\mu \omega_\mu, y^b\} &= \delta_a^b, \\ \{p_a^\mu \omega_\mu, y^b \omega_\nu\} &= \delta_a^b \omega_\nu, \\ \{p_a^\mu, y^b \omega_\nu\} &= \delta_a^b \delta_\nu^\mu \end{aligned} \quad (2.3)$$

and are seen to reduce to the canonical Poisson bracket in mechanics when $n = 1$. Hence, the pairs of variables entering the brackets (2.3) can be viewed as canonically conjugate with respect to the graded Poisson bracket (2.2). Note that no dependence on x^μ -s is implied in (2.3) so that the above brackets should be viewed rather as “equal-point”, as opposite to the conventional “equal-time” ones.

Now, by considering the brackets of canonical variables with H or $H\omega$ one can write the DW canonical field equations (1.1) in graded Poisson bracket formulation, e.g.

$$\begin{aligned} \mathbf{d}(y^a \omega_\mu) &= \{[H\omega, y^a \omega_\mu]\}, \\ \mathbf{d}(p_a^\mu \omega_\mu) &= \{[H\omega, p_a^\mu \omega_\mu]\}, \end{aligned} \quad (2.4)$$

where \mathbf{d} is the total exterior differential such that, for example, $\mathbf{d}y = \partial_\mu y(x) dx^\mu$. The DW canonical equations written in the form (2.4) point to the fact that the type of the space-time variations controlled by H is intimately related to the exterior differentiation. This generalizes to the present formulation of field theory the familiar statement that Hamiltonian generates a time evolution. Note that the form (2.4) of the canonical field equations underlies our hypothesis (2.6) regarding the form of a generalized Schrödinger equation within the pre-canonical approach (see [15, 28, 29]).

2.2 Quantization

Quantization of a Gerstenhaber algebra \mathcal{G} or its above-mentioned generalizations would be a difficult mathematical problem. We may even need to modify the notion of quantization or deformation to treat this problem properly [31]. This is due to the fact that $bdeg(a) \neq pdeg(a)$ for $a \in \mathcal{G}$. An attempt to adopt geometric quantization to the present case also faces this problem already on the pre-quantization level. Fortunately, in physics we usually do not need to quantize the whole Poisson algebra. It is even impossible in the sense of canonical quantization, as it follows from the Groenewold-van Hove “no-go” theorem [32, 33]. In fact, quantization of a small Heisenberg subalgebra of the canonical brackets often suffices. All we need to know about the rest of a Poisson algebra is essentially that “it is there”.

Thus, as the first step it seems reasonable to restrict our attention to a small subalgebra of graded Poisson brackets which is similar to the Heisenberg subalgebra of a Poisson algebra in mechanics. A natural candidate is the subalgebra of canonical brackets in the Lie subalgebra of Hamiltonian forms of degree 0 and $(n - 1)$, see eqs. (2.3). The scheme of field quantization discussed in [15, 28, 29] is essentially based on quantization of this small subalgebra by the Dirac correspondence rule. The following realization of operators corresponding to the quantities involved in (2.3) was proposed

$$\begin{aligned}\widehat{p_a^\mu \omega_\mu} &= i\hbar \partial/\partial y^a, \\ \hat{p}_a^\nu &= -i\hbar \kappa \gamma^\nu \partial/\partial y^a, \\ \hat{\omega}_\nu &= -\kappa^{-1} \gamma_\nu,\end{aligned}\tag{2.5}$$

where γ^μ are the imaginary units of the Clifford algebra of the space-time manifold and the parameter κ of the dimension $(\text{length})^{-(n-1)}$ is required by the dimensional consistency of (2.5). A possible identification of κ with the ultra-violet cutoff or the fundamental length scale quantity was discussed in [28, 30]. Note, that realization (2.5) is essentially inspired by the relation between the Clifford algebra and the endomorphisms of the exterior algebra. A crucial assumption underlying the proof that operators in (2.5) fulfill the commutators following from (2.3) is that the composition law of operators implies the symmetrized product of γ -matrices.

The realization (2.5) suggests that quantization of DW formulation, viewed as a multi-parameter generalization of the standard Hamiltonian formulation with a single time parameter, results in a version, or a generalization, of the quantum theoretic formalism in which the hypercomplex (Clifford) algebra of the underlying space-time manifold generalizes the algebra of complex numbers (=the Clifford algebra of $(0+1)$ -dimensional “space-time”) in quantum mechanics, and in which n space-time variables being treated on equal footing generalize the single time parameter. In doing so quantum mechanics appears as a special case corresponding to $n = 1$. This philosophy suggests the following generalization of the Schrödinger equation to the present framework [28–30]

$$i\hbar \kappa \gamma^\mu \partial_\mu \Psi = \widehat{H} \Psi,\tag{2.6}$$

where \widehat{H} is the operator corresponding to the DW Hamiltonian function, the constant κ of the dimension $(\text{length})^{-(n-1)}$ appears again on dimensional grounds, and $\Psi = \Psi(y^a, x^\mu)$ is the wave function over the covariant configuration space of field and space-time variables.

The generalized Schrödinger equation (2.6) satisfies several aspects of the correspondence principle [29, 30]. In particular, it leads, at least in the simplest case of scalar fields, to the DW canonical equations (1.1) for the mean values of appropriate operators (the Ehrenfest theorem) and reduces to the DW Hamilton-Jacobi equation (1.2) (with some additional conditions) in the classical limit. For an application of the presented scheme to the case of scalar fields see [29, 30].

3 Quantizing General Relativity

In this section we first outline curved space-time generalization of the scheme presented in sect. 2.2 and then discuss its further application to quantization of General Relativity. The latter requires the DW Hamiltonian formulation of General Relativity which is discussed in sect. 3.2. This framework leads to a covariant Schrödinger equation which is supposed to describe quantum General Relativity and a supplementary condition which introduces a background geometry in a self-consistent with the underlying quantum dynamics way.

3.1 Curved space-time generalization

To apply the above framework to General Relativity we first need to extend it to curved space-time with metric $g_{\mu\nu}(x)$. The extension of the generalized Schrödinger equation (2.6) to curved space-time is similar to that of the Dirac equation, i.e.

$$i\hbar\kappa\gamma^\mu(x)\nabla_\mu\Psi = \widehat{H}\Psi \quad (3.1)$$

where \widehat{H} is an operator form of the DW Hamiltonian function and ∇_μ is the covariant derivative, $\nabla_\mu := \partial_\mu + \theta_\mu(x)$. We introduced x -dependent γ -matrices which fulfill

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2g_{\mu\nu}(x) \quad (3.2)$$

and can be expressed with the aid of vielbein fields $e_\mu^A(x)$, such that

$$g_{\mu\nu}(x) = e_\mu^A(x)e_\nu^B(x)\eta_{AB}, \quad (3.3)$$

and the Minkowski space Dirac matrices γ^A , $\gamma^A\gamma^B + \gamma^B\gamma^A := 2\eta^{AB}$,

$$\gamma^\mu(x) := e_A^\mu(x)\gamma^A.$$

If Ψ is a spinor then $\nabla_\mu = \partial_\mu + \theta_\mu$ is the spinor covariant derivative with

$$\theta_\mu = \frac{1}{4}\theta_{AB\mu}\gamma^{AB},$$

$\gamma^{AB} := \frac{1}{2}(\gamma^A\gamma^B - \gamma^B\gamma^A)$, denoting the spinor connection whose components are known to be given by

$$\theta^A{}_{B\mu} = e_\alpha^A e_B^\nu \Gamma^\alpha{}_{\mu\nu} - e_B^\nu \partial_\mu e_\nu^A. \quad (3.4)$$

For example, interacting scalar fields ϕ^a on a curved background are described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\{\partial_\mu\phi^a\partial^\mu\phi_a - U(\phi^a) - \xi R\phi^2\},$$

where $g := |\det(g_{\mu\nu})|$. It gives rise to the following expressions of polymomenta and the DW Hamiltonian density

$$p_a^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} = \sqrt{g}\partial^\mu\phi_a, \quad \sqrt{g}H = \frac{1}{2\sqrt{g}}p_a^\mu p_\mu^a + \frac{1}{2}\sqrt{g}\{U(\phi) + \xi R\phi^2\}$$

for which the corresponding operators can be found to take the form

$$\begin{aligned}\widehat{p}_a^\mu &= -i\hbar\kappa\sqrt{g}\gamma^\mu\frac{\partial}{\partial\phi^a}, \\ \widehat{H} &= -\frac{\hbar^2\kappa^2}{2}\frac{\partial^2}{\partial\phi^a\partial\phi_a} + \frac{1}{2}\{U(\phi) + \xi R\phi^2\}.\end{aligned}\quad (3.5)$$

3.2 Pre-canonical approach to quantum General Relativity

In the context of General Relativity viewed as a field theory the metric $g_{\alpha\beta}$ (or the vielbein e_A^μ) is the field variable. Hence, according to the pre-canonical scheme, the wave function is a function of space-time and metric (or vielbein) variables, i.e. $\Psi = \Psi(x^\mu, g^{\alpha\beta})$ (or $\Psi = \Psi(x^\mu, e_A^\mu)$). To formulate an analogue of the Schrödinger equation for this wave function we need γ -matrices which fulfill

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \quad (3.6)$$

and can be related to the Minkowski space γ -matrices γ^A by means of the vielbein components: $\gamma^\mu := e_A^\mu\gamma^A$, where $g^{\mu\nu} := e_A^\mu e_B^\nu \eta^{AB}$. Note that, as opposite to the theory on curved background, no explicit dependence on space-time variables is present in the above formulas; instead, e_A^μ , γ^μ and $g^{\mu\nu}$ are viewed as the fibre coordinates in the corresponding bundles over the space-time.

Now, modelled after (3.1), the following (symbolic form of the) generalized Schrödinger equation for the wave function of quantized gravity can be put forward

$$i\hbar\kappa e\widehat{\nabla}\Psi = \widehat{\mathcal{H}}\Psi, \quad (3.7)$$

where $\widehat{\mathcal{H}}$ is the operator form of the DW Hamiltonian density of gravity, $\widehat{\mathcal{H}} := e\widehat{H}$, an explicit form of which remains to be constructed, $e := |\det(e_\mu^A)|$, and $\widehat{\nabla}$ denotes the quantized Dirac operator in the sense that the corresponding connection coefficients are replaced by appropriate differential operators (c.f., e.g., eqs. (3.15), (3.16) and (3.21) below). Note also, that in the context of quantum gravity it seems to be very natural to identify the parameter κ in (3.7) with the Planck scale quantity, i.e. $\kappa \sim l_{\text{Planck}}^{-(n-1)}$.

If the wave function in (3.7) is spinor then the covariant derivative operator $\widehat{\nabla}_\mu$ contains the spinor connection which on the classical level involves the term with space-time derivatives of vielbeins (c.f. eq. (3.4)) which cannot be expressed in terms of quantities of the metric formulation. Consequently, the spinor nature of equation (3.7) necessitates the use of the vielbein formulation of General Relativity. However, since no suitable DW formulation of General Relativity in vielbein variables is available so far, the subsequent consideration will be based on the metric formulation while the analysis based on the vielbein formulation is postponed to a future publication.

3.2.1 DW formulation of the Einstein equations

A suitable DW-like formulation of General Relativity in metric variables was presented earlier by Hořava [35]. In this formulation field variables are chosen to be the metric

density components $h^{\alpha\beta} := \sqrt{g}g^{\alpha\beta}$ and the role of polymomenta is taken on by the following combination of the Christoffel symbols

$$Q_{\beta\gamma}^\alpha := \frac{1}{8\pi G}(\delta_{(\beta}^\alpha \Gamma_{\gamma)\delta}^\delta - \Gamma_{\beta\gamma}^\alpha). \quad (3.8)$$

Respectively, the DW Hamiltonian density $\mathcal{H} := \sqrt{g}H$ assumes the form

$$\mathcal{H}(h^{\alpha\beta}, Q_{\beta\gamma}^\alpha) := 8\pi G h^{\alpha\gamma} \left(Q_{\alpha\beta}^\delta Q_{\gamma\delta}^\beta + \frac{1}{1-n} Q_{\alpha\beta}^\beta Q_{\gamma\delta}^\delta \right) + (n-2)\Lambda\sqrt{g} \quad (3.9)$$

which is essentially the truncated Lagrangian density of General Relativity written in terms of variables $h^{\alpha\beta}$ and $Q_{\beta\gamma}^\alpha$.

Using these variables the Einstein field equations are formulated in DW Hamiltonian form as follows

$$\partial_\alpha h^{\beta\gamma} = \partial\mathcal{H}/\partial Q_{\beta\gamma}^\alpha, \quad (3.10)$$

$$\partial_\alpha Q_{\beta\gamma}^\alpha = -\partial\mathcal{H}/\partial h^{\beta\gamma}, \quad (3.11)$$

where eq. (3.10) is equivalent to the well-known expression of the Christoffel symbols in terms of the metric while eq. (3.11) yields the vacuum Einstein equations in terms of the Christoffel symbols.

3.2.2 Naive pre-canonical quantization

Now, to quantize General Relativity in DW formulation we can *formally* follow the curved space-time version of our scheme, sect. 3.1. This leads to the following operator form of polymomenta $Q_{\beta\gamma}^\alpha$

$$\hat{Q}_{\beta\gamma}^\alpha = -i\hbar\kappa\gamma^\alpha \left\{ \sqrt{g} \frac{\partial}{\partial h^{\beta\gamma}} \right\}_{ord} \quad (3.12)$$

up to an ordering ambiguity of the expression inside the curly brackets. By substituting this expression to (3.9) and performing some manipulations using the assumption of the “standard” ordering (that differential operators are all collected to the right) and relation (3.6) for curved γ -matrices we obtain the following operator form of the DW Hamiltonian density

$$\widehat{\mathcal{H}} = -8\pi G \hbar^2 \kappa^2 \frac{n-2}{n-1} \left\{ \sqrt{g} h^{\alpha\gamma} h^{\beta\delta} \frac{\partial}{\partial h^{\alpha\beta}} \frac{\partial}{\partial h^{\gamma\delta}} \right\}_{ord} + (n-2)\Lambda\sqrt{g} \quad (3.13)$$

However, it should be pointed out that the above procedure is rather of heuristic nature and requires a mathematical justification. Namely, according to (3.8) classical polymomenta $Q_{\beta\gamma}^\alpha$ transform as connection coefficients while the operator we associated with them in (3.12) is a tensor. Moreover, the classical DW Hamiltonian density in (3.9) does not transform as a scalar density, while the operator constructed in (3.13) is a scalar density. It is thus natural to inquire whether, or in which sense, the whole procedure is meaningful.

3.2.3 Covariant Schrödinger equation for quantized gravity and the “bootstrap condition”

The answer to the above question refers to the observation that canonical quantization procedures are in fact generally performed with respect to a specific reference frame and require “covariantization” as a subsequent step. Note now, that the consistency of expressions (3.12) and (3.13) with classical transformation laws could be achieved by adding an auxiliary term in (3.12) which transforms as a connection and can be interpreted as appearing due to an auxiliary connection corresponding to a chosen coordinate system or a background. Then, using the expression of the Christoffel symbols in terms of the polymomenta $Q_{\beta\gamma}^\alpha$ (c.f. (3.8))

$$\Gamma_{\beta\gamma}^\alpha = 8\pi G \left(\frac{2}{n-1} \delta_{(\beta}^\alpha Q_{\gamma)\delta}^\delta - Q_{\beta\gamma}^\alpha \right) \quad (3.14)$$

we are led to the following ordering dependent operator form of the Christoffel symbols

$$\hat{\Gamma}_{\beta\gamma}^\alpha = -8\pi i G \hbar \kappa \sqrt{g} \left(\frac{2}{n-1} \delta_{(\beta}^\alpha \gamma^\sigma \frac{\partial}{\partial h^{\gamma)\sigma}} - \gamma^\alpha \frac{\partial}{\partial h^{\beta\gamma}} \right) + \tilde{\Gamma}_{\beta\gamma}^\alpha(x) \quad (3.15)$$

where the auxiliary connection is denoted as $\tilde{\Gamma}_{\beta\gamma}^\alpha(x)$. However, if we keep the arbitrary term $\tilde{\Gamma}_{\beta\gamma}^\alpha(x)$ it will enter the final results thus leading to a rather unappealing background dependent theory.

On the other hand, we can notice that our operators essentially arise from the “equal-point” commutation relations (c.f. eqs. (2.3)) and thus can be viewed as locally defined “in a point”. Now, in an infinitesimal vicinity of a point we always can chose a local coordinate system in which the auxiliary connection $\tilde{\Gamma}_{\beta\gamma}^\alpha(x)$ vanishes and then think of this system as the one which is actually meant when writing expression (3.12) for operators \hat{Q} . This, however, when consistently implemented, requires a subsequent “patching together” procedure which is likely to lead to extra terms in our generalized Schrödinger equation due to the connection involved in “patching together”.

To cope with the above-mentioned problems we proceed as follows. Let us first write the generalized Schrödinger equation (3.7) in the local coordinate system in the vicinity of a point x , in which $\tilde{\Gamma}_{\beta\gamma}^\alpha|_x = 0$, and then covariantize it in the simplest way. The first step leads to the locally valid equation

$$i\hbar\kappa\sqrt{g}\gamma^\mu(\partial_\mu + \hat{\theta}_\mu)\Psi = \hat{\mathcal{H}}\Psi. \quad (3.16)$$

Next, the operator form of the spinor connection coefficients, as follows from (3.4) and (3.15), is given by

$$\begin{aligned} \hat{\theta}^A_{B\mu} &= -8\pi i G \hbar \kappa \sqrt{g} e_\alpha^A e_B^\nu \left(\frac{2}{n-1} \delta_{(\mu}^\alpha \gamma^\sigma \frac{\partial}{\partial h^{\nu)\sigma}} - \gamma^\alpha \frac{\partial}{\partial h^{\mu\nu}} \right) + \tilde{\theta}^A_{B\mu}(x) \\ &=: (\theta^A_{B\mu})^{op} + \tilde{\theta}^A_{B\mu}(x) \end{aligned} \quad (3.17)$$

where $(\theta^A_{B\mu})^{op}$ is a notation introduced for the first ordering dependent operatorial term. Note that in general $\tilde{\theta}^A_{B\mu}|_x \neq 0$ even though $\tilde{\Gamma}_{\beta\gamma}^\alpha|_x = 0$.

Now, to formulate a covariant version of (3.16) we notice that vielbeins do not enter the present DW Hamiltonian formulation of General Relativity and, therefore, within the present consideration may (and can only) be treated as non-quantized classical x -dependent quantities describing a reference vielbein field which accounts for a choice of coordinates or a background.

On another hand, the bilinear combination of vielbeins $e_A^\mu e_B^\nu \eta^{AB}$ is the metric $g^{\mu\nu}$ which is a canonical variable quantized as an x -independent quantity (in the “ultra-Schrödinger” picture used here, in which all space-time dependence is converted to the wave function while the operators are space-time independent). Both aspects can be reconciled by requiring that the bilinear combination of vielbeins should be consistent with the mean value of the metric, i.e.

$$\tilde{e}_A^\mu(x) \tilde{e}_B^\nu(x) \eta^{AB} = \langle g^{\mu\nu} \rangle(x), \quad (3.18)$$

where the latter is given by averaging over the space of the metric components by means of the wave function $\Psi(g^{\mu\nu}, x^\mu)$:

$$\langle g^{\mu\nu} \rangle(x) = \int [dg^{\alpha\beta}] \bar{\Psi}(g, x) g^{\mu\nu} \Psi(g, x). \quad (3.19)$$

The invariant integration measure in (3.19) can be found to be (c.f. [36])

$$[dg^{\alpha\beta}] = \sqrt{g}^{(n+1)} d^{\frac{1}{2}n(n+1)} g^{\alpha\beta}. \quad (3.20)$$

Thus, the background metric geometry explicitly appears as a result of quantum averaging of the metric operator $g^{\mu\nu}$, while the local orientation of vielbeins is supposed to be exclusively due to a choice of a reference vielbein field (local reference frames) on the macroscopic level, which is restricted only by the consistency with the averaged metric according to the “bootstrap condition” (3.18).

Now, the covariantized version of (3.16) can be written in the form

$$i\hbar\kappa\tilde{e}_A^\mu(x)\gamma^A(\partial_\mu + \tilde{\theta}_\mu(x))\Psi + i\hbar\kappa(\sqrt{g}\gamma^\mu\theta_\mu)^{op}\Psi = \hat{\mathcal{H}}\Psi. \quad (3.21)$$

The explicit form of the term $(\sqrt{g}\gamma^\mu\theta_\mu)^{op}$ can be derived from (3.17) by assuming the “standard” ordering in the intermediate calculations and replacing the appearing therewith bilinear combinations of vielbeins with the metric. This procedure yields the result

$$(\sqrt{g}\gamma^\mu\theta_\mu)^{op} = -n\pi i G \hbar \kappa \left\{ \sqrt{g} h^{\mu\nu} \frac{\partial}{\partial h^{\mu\nu}} \right\}_{ord} \quad (3.22)$$

which is also ordering dependent. The x -dependent reference spinor connection term $\tilde{\theta}_\mu(x)$ in (3.21) is calculated using the reference vielbein field $\tilde{e}_\mu^A(x)$ consistent with (3.18) and the classical expression

$$\tilde{\theta}_\mu^{AB}(x) = \tilde{e}^{\alpha[A} \left(\frac{1}{2} \partial_{[\mu} \tilde{e}_{\alpha]}^{B]} + \tilde{e}^{B]\beta} \tilde{e}_\mu^C \partial_\beta \tilde{e}_{C\alpha} \right) \quad (3.23)$$

which is equivalent to (3.4). Lastly, to distinguish a physically relevant information in (3.21) we need to impose a gauge-type condition on Ψ which, in the case of the De Donder-Fock harmonic gauge, can be written in the form

$$\partial_\mu \langle \sqrt{g} g^{\mu\nu} \rangle (x) = 0. \quad (3.24)$$

Thus, we conclude that within the quantization based on DW formulation the quantized gravity may be described by the generalized Schrödinger equation (3.21), with operators $\widehat{\mathcal{H}}$ and $(\sqrt{g}\gamma^\mu\theta_\mu)^{op}$ given respectively by (3.13) and (3.22), and the supplementary “bootstrap condition” (3.18). These equations in principle allow us to obtain the wave function $\Psi(g^{\mu\nu}, x^\alpha)$ which then may be interpreted as the probability amplitude to find the values of the components of the metric in the interval $[g^{\mu\nu} - (g^{\mu\nu} + dg^{\mu\nu})]$ in an infinitesimal vicinity of the point x^α . Obviously, this description is very different from the conventional quantum field theoretic one and its physical significance remains to be investigated. Note, however, that it opens an intriguing possibility to approximate the “wave function of the Universe” by the fundamental solution of equation (3.21). This solution is indeed expected to describe an expansion of the wave function from the primary “probability clot” of the Planck scale and assigns a meaning to the “genesis of the space-time” in the sense that the observation of the space-time points beyond the primary “clot” becomes more and more probable with the spreading of the wave function. The self-consistency encoded in the “bootstrap condition” plays a crucial role in such a “genesis”: in a sense, the wave function itself determines, or “lays down”, the space-time geometry it is to propagate on.

4 Conclusion

We discussed an application of quantization of fields based on the De Donder-Weyl canonical theory, viewed as a manifestly covariant generalization of the Hamiltonian formulation from mechanics to field theory, to the problem of quantization of gravity.

The analogue of the Hamiltonian formulation which underlies the present procedure of quantization does not need a distinction between space and time dimensions for, it treats fields as systems *varying* in space-time rather than those *evolving* in time. The De Donder-Weyl canonical equations (1.1) describe this type of varying in space-and-time. As a result, the quantum counterpart of the theory is formulated on a finite dimensional configuration space of space-time and field variables, with the corresponding wave *function* $\Psi(x^\mu, y^a)$ naturally interpreted as the probability amplitude of a field to have values in the interval $[y - (y + dy)]$ in the vicinity of the space-time point x . Correspondingly, all the dependence on a space-time location is transferred from operators to the wave function, corresponding to what could be called the “ultra-Schrödinger” picture.

One of the unsolved problems of the present approach to be mentioned is its connection with the contemporary notions of quantum field theory. The lack of a proper understanding of this issue so far has been preventing specific applications of the present framework (see, however, [39] for a recent attempt to apply it to quantization of p -

branes). Nevertheless, one can hope that the already understood character of connections between the De Donder-Weyl canonical theory and the standard Hamiltonian formalism may help us to clarify this problem. Namely, the De Donder-Weyl formulation in a sense is an intermediate step between the manifestly covariant Lagrangian formulation and the canonical Hamiltonian framework [11, 37]. In particular, the standard symplectic form and the standard equal-time canonical brackets in field theory can be obtained by the integration of the polysymplectic form Ω and the canonical brackets (2.3) over the Cauchy data surface in the covariant configuration space (x^μ, y^a) [16]. Similarly, the standard functional derivative field theoretic Hamilton and Hamilton-Jacobi equations can be deduced from the DW Hamiltonian and the DW Hamilton-Jacobi equations. The corresponding derivations involve a pull-back of the quantities of DW formulation to a Cauchy data surface Σ given by $(y^a = y_{in}^a(\mathbf{x}), t = t_{in})$ and a subsequent integration over it. It is natural to inquire if similar connections can be established between the elements of the present approach to field quantization and those of the standard canonical quantization.

Another possible way to establish a connection with the conventional QFT is to view the Schrödinger wave functional $\Psi([y(\mathbf{x})], t)$ [38] as a composition of amplitudes given by our wave functions $\Psi(y, \mathbf{x}, t)$. In [30] we discussed this connection in the ultra-local approximation but its extension beyond this approximation so far remains problematic.

The character of the relationships between the DW formulation and the conventional canonical formalism allows us to view the former as what could be called a *pre-canonical* formalism. The term reflects the intermediate position of the latter between the covariant Lagrangian and the canonical Hamiltonian levels of description. Note that in mechanics ($n = 1$) pre-canonical description coincides with the canonical one while in field theory ($n > 1$) they become different. Hence the questions naturally arise as to what would be a pre-canonical analogue of quantization and what is the physical significance of the corresponding *pre-canonical quantization*. The present paper can be viewed as an attempt to shed some light on these questions.

The application of the present (pre-canonical) framework to gravity immediately poses many questions to which no final answers can be given as yet. Some of these, such as, e.g., (i) how the spinor wave function can be reconciled with the boson vs. fermion nature of the fields we are about to quantize, (ii) if it can or should be replaced by a more general Clifford algebra valued wave function, (iii) to which extent one can trust to the prescription (3.19) of quantum averaging of operators notwithstanding the underlying scalar product is neither positive definite nor space-time location independent, (iv) how to quantize operators more general than those entering the pre-canonical brackets (2.3), and, at last, (v) how to calculate observable quantities of interest in field theory using the present framework, concern rather the pre-canonical approach in general and we hope to address them in future publications. Let us instead concentrate here on a few questions related to the specific application to General Relativity.

One of the severe problems we encountered is due to, on the one hand, the non-tensorial nature of basic quantities (polymomenta and the DW Hamiltonian) of the present DW formulation of General Relativity, which is based in essence on the trun-

cated Lagrangian density containing no second-order derivatives of the metric, instead of the generally covariant density $\sqrt{q}R$, and, on the other hand, the tensorial character of operators which only can be constructed as quantum counterparts of these quantities. To avoid the problem we adopted the concept of quantization in the vicinity of a point and subsequent covariantization. This procedure, however, involves external elements, such as the reference or background vielbein field $\tilde{e}_a^\mu(x)$ and the corresponding spinor connection $\tilde{\theta}_\mu(x)$, which enter as non-quantized entities into the generalized Schrödinger equation (3.21). To make the theory self-consistent we introduced a supplementary “bootstrap condition” (3.18) which connects the bilinear combination of vielbein fields $\tilde{e}_a^\mu(x)$ with the quantum mean value of the metric. By this means the allowable classical geometrical background is included into the theory in a self-consistent with the underlying quantum dynamics way. Clearly, this point of view is much less radical than the usual denial of any background geometrical structure which, in fact, is the source of most of the conceptual difficulties of quantum gravity. However, within the present scheme it seems to offer an alternative to various “pre-geometrical” constructions.

Next, it should be pointed out that the coefficients involving n in (3.13) and (3.22) at the present stage cannot be considered as reliably established. This is related both to the ordering ambiguity and the unreliability of results obtained by formal substitution of polymomenta operators (3.12) to classical expressions (for example, applying the similar procedure to the DW Hamiltonian of a massless scalar field y yields the operator $-\frac{n}{2}\hbar^2\kappa^2\partial_{yy}^2$ instead of the correct one $-\frac{1}{2}\hbar^2\kappa^2\partial_{yy}^2$ [28, 29]). Let us note also, that at this stage it is rather difficult to choose between the formulation based on the operator of DW Hamiltonian \widehat{H} and that based on the corresponding density $\widehat{\mathcal{H}}$. In the former case, the generalized Schrödinger equation is modified as follows: $i\hbar\kappa\gamma^\mu\widehat{\nabla}_\mu\Psi = \widehat{H}\Psi$ which in general is different from (3.7) due to the ordering ambiguity. A preliminary consideration of toy one-dimensional models corresponding to the formulations using respectively \widehat{H} and $\widehat{\mathcal{H}}$ indicates that the latter formulation, which leads to a toy model similar to that discussed long ago by Klauder [40], reveals more interesting behavior and thus may be considered as more suitable. To present more conclusive results, however, an additional analysis, possibly based on quantization of more general Poisson brackets than pre-canonical ones in (2.3), is required.

Moreover, as we have already pointed out, the vielbein formulation of General Relativity is a more suitable starting point for the application of pre-canonical quantization to gravity. The corresponding analysis is in progress and we hope to present it elsewhere. Needless to add that much work has to be done to make the present approach comparable with other developments in quantum theory of gravity.

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REFERENCES

- [1] Rovelli C 1998 Strings, loops and others: a critical survey of the present approaches to quantum gravity **gr-qc/9803024**
- [2] Isham C 1995 Structural Issues in Quantum Gravity **gr-qc/9510063**
Isham C 1992 Canonical Quantum Gravity and the Problem of Time **gr-qc/9210011**
- [3] Donoghue J F 1996 *Helv. Phys. Acta* **69** 269- 75 **gr-qc/9607039**
Donoghue J F 1994 *Phys.Rev.* **D50** 3874-88 **gr-qc/9405057**
- [4] De Donder Th 1935 *Theorie Invariantive du Calcul des Variations* (Paris: Gauthier-Villars)
Weyl H. 1935 Geodesic fields in the calculus of variations *Ann. Math. (2)* **36** 607-29
For a review see: Rund H 1966 *The Hamilton-Jacobi Theory in the Calculus of Variations* (Toronto: D. van Nostrand), Kastrup H 1983 Canonical theories of Lagrangian dynamical systems in physics *Phys. Rep.* **101** 1-167
- [5] Dickey L A 1994 Field-theoretical (multi-time) Lagrange-Hamilton formalism and integrable equations in: *Lectures on Integrable Systems, In Memory of Jean-Louis Verdier* eds. Babelon O et al (Singapore: World Scientific) p. 103-62
- [6] von Rieth J 1984 The Hamilton-Jacobi theory of De Donder and Weyl applied to some relativistic field theories *J. Math. Phys.* **25** 1102-15
- [7] Dedecker P 1977 On the generalizations of symplectic geometry to multiple integrals in the calculus of variations *Lect Notes Math* vol **570** (Berlin: Springer) p. 395-456
- [8] Krupka D 1983 Lepagean forms in high order variational theory in: *Proc IUTAM-ISIMM Symp on Modern Developments in Analytical Mechanics* vol 1, Atti Accad Sci Torino Suppl vol **117** p. 197-238
Krupka D 1986 Regular Lagrangians and Lepagean forms in: *Differential Geometry and its Applications* (Proc. Conf., Brno, Czechoslovakia, Aug 24-30 1986) p. 111-148
Krupka D 1987 Geometry of Lagrangean structures 3 *Suppl Rend Circ Mat Palermo* Ser II No 14 187-224
- [9] Gotay M J 1991 An exterior differential systems approach to the Cartan form in: *Symplectic Geometry and Mathematical Physics* eds. Donato P et al (Boston: Birkhäuser) p. 160-88
- [10] Good R H jr 1954 Hamiltonian mechanics of fields *Phys. Rev.* **93** 239-43
Edelen D G B 1961 The invariance group for Hamiltonian systems of partial differential equations *Arch. Rat. Mech. Anal.* **5** 95-176
Hermann R 1970 *Lie Algebras and Quantum Mechanics* (New York: W.A.Benjamin, inc.)
Marsden J E Montgomery R Morrison P J and Thompson W P 1986 Covariant Poisson bracket for classical fields *Ann. Phys.* **169** 29-47
Tapia V 1988 Covariant field theory and surface terms *Nuovo Cim.* **102 B** 123-30
- [11] Goldschmidt H and Sternberg S 1973 The Hamilton-Cartan formalism in the calculus of variations *Ann. Inst. Fourier (Grenoble)* **23** 203-67

- [12] Cariñena J E Crampin M and Ibort L A 1991 On the multisymplectic formalism for first order field theories *Diff. Geom. and its Appl.* **1** 345-74
- [13] Gotay M J Isenberg J and Marsden J 1998 *Momentum Maps and Classical Relativistic Fields* (Berkeley preprint, various versions exist since 1985) Part I: Covariant Field Theory [physics/9801019](#)
- [14] Günther C 1987 The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case *J. Diff. Geom.* **25** 23-53
- [15] Kanatchikov I V 1995 From the Poincaré-Cartan form to a Gerstenhaber algebra of Poisson brackets in field theory in: *Coherent States, Quantizations, and Complex Structures* eds. Antoine J-P et al (New York: Plenum Press) p. 173-83 [hep-th/9511039](#)
- [16] Kanatchikov I V 1998 Canonical structure of classical field theory in the polymomentum phase space *Rep. Math. Phys.* **41** 49-90 [hep-th/9709229](#)
- [17] Kanatchikov I V 1997a On field theoretic generalizations of a Poisson algebra *Rep. Math. Phys.* **40** 225-234 [hep-th/9710069](#)
 Kanatchikov I V 1997b Novel algebraic structures from the polysymplectic form in field theory in: *GROUP21, Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras* vol. 2 eds. Doebner H-D et al (Singapore: World Scientific) p. 894-9 [hep-th/9612255](#)
- [18] Echeverría-Enríquez A and Muñoz-Lecanda M C 1992 Variational calculus in several variables: a hamiltonian approach *Ann. Inst. Henri Poincaré* **56** 27-47
 Echeverría-Enríquez A Muñoz-Lecanda M C and Roman-Roy N 1996 Geometry of Lagrangian First-order Classical Field Theories, *Fortsch. Phys.* **44** 235-80, [dg-ga/9505004](#)
 Echeverría-Enríquez A Muñoz-Lecanda M C and Roman-Roy N 1997 Multivector Fields and Connections. Setting Lagrangian Equations in Field Theories *J. Math. Phys.* **39** 4578-603 [dg-ga/9707001](#)
 Ibort A Echeverría-Enríquez A Muñoz-Lecanda M C and Roman-Roy N 1998 Invariant Forms and Automorphisms of Multisymplectic Manifolds [math-ph/9805040](#)
 Ibort A Echeverría-Enríquez A Muñoz-Lecanda M C and Roman-Roy N 1999a On the Multimomentum Bundles and the Legendre Maps in Field Theories [math-ph/9904007](#) to appear in *Rep. Math. Phys.*
 Echeverría-Enríquez A Muñoz-Lecanda M C and Roman-Roy N 1999b Multivector field formulation of Hamiltonian field theories: equations and symmetries [math-ph/9907007](#)
- [19] Giachetta G Mangiarotti L and Sardanashvily G 1997 *New Lagrangian and Hamiltonian Methods in Field Theory* (Singapore: World Scientific)
- [20] de León M Marín-Solano J and Marrero J C 1995 Ehresmann connections in classical field theories, *Anales de Física, Monografías* **2** 73-89
 de León M Marín-Solano J and Marrero J C 1996 A geometrical approach to classical field theories: a constraint algorithm for singular theories in: *New Developments in Differential Geometry* eds. Tamassi L and Szenthe J (Amsterdam: Kluwer) p. 291-312
- [21] Marsden J and Shkoller S 1998 Multisymplectic geometry, covariant hamiltonians, and water waves to appear in *Math. Proc. Camb. Phil. Soc.* [math.DG/9807086](#)
- [22] Hrabak S P 1999a,b On a multisymplectic formulation of the classical BRST symmetry for first order field theories Part 1: Algebraic structures [math-ph/9901012](#), Part 2: Geometric structures [math-ph/9901013](#)

- [23] Cantrijn F Ibort L A and de Leon M 1999 On the geometry of multisymplectic manifolds *J. Australian Math. Soc.* (Ser. A) **66** 303-30
 Cantrijn F Ibort L A and de Leon M 1996 Hamiltonian structures on multisymplectic manifolds *Rend. Sem. Mat. Univ. Pol. Torino* **54**(pt I) 225-36
- [24] Fulp R O Lawson J K and Norris L K 1996 Generalized symplectic geometry as a covering theory for the Hamiltonian theories of classical particles and fields *J. Geom. Phys.* **20** 195-206
 Lawson J K 1997 A frame bundle generalization of multisymplectic field [dg-ga/9706008](#)
 D. Cartin D 1997 Generalized symplectic manifolds [dg-ga/9710027](#)
- [25] Born M 1934 On the quantum theory of the electromagnetic field *Proc. Roy. Soc. (London)* **A143** 410-37
 Weyl H 1934 Observations on Hilbert's independence theorem and Born's quantization of field equations *Phys. Rev.* **46** 505-8
- [26] Günther C 1987 Polysymplectic quantum field theory in: *Differential Geometric Methods in Theoretical Physics* (Proc. XV Int Conf) eds. Doebner H-D and Hennig J D (Singapore: World Scientific) p. 14-27
 Good R H 1994 Mass spectra from field equations I *J. Math. Phys.* **35** 3333-9
 Good R H 1995 Mass spectra from field equations II *J. Math. Phys.* **36** 707-13
 Navarro M 1995 Comments on Good's proposal for new rules of quantization *J. Math. Phys.* **36** 6665-72 [hep-th/9503068](#)
 Sardanashvily G 1994 Multimomentum Hamiltonian formalism in quantum field theory *Int. J. Theor. Phys.* **33** 2365-2380 [hep-th/9404001](#)
 Navarro M 1998 Toward a finite dimensional formulation of quantum field theory *Found Phys Lett* **11** 585-93 [quant-ph/9805010](#)
- [27] Loday J-L 1993 Une version non commutative des algèbres de Lie. Les algèbres de Leibniz *L'Enseign. Math.* **39** 269-93
- [28] Kanatchikov I V 1998 Toward the Born-Weyl quantization of fields *Int. J. Theor. Phys.* **37** 333-42 [quant-ph/9712058](#)
- [29] Kanatchikov I V 1999 De Donder-Weyl theory and a hypercomplex extension of quantum mechanics to field theory *Rep. Math. Phys.* **43** 157-70 [hep-th/9810165](#)
- [30] Kanatchikov I V 1998 On quantization of field theories in polymomentum variables in: *Particles, Fields and Gravitation* (Proc. Int. Conf. Lodz, Poland, Apr. 1998) ed. Rembielinski J *AIP Conf. Proc.* vol. **453** p. 356-67 [hep-th/9811016](#)
- [31] Flato M 1997 private communication
- [32] Emch G G 1984 *Mathematical and Conceptual Foundations of 20th-Century Physics* (Amsterdam: North-Holland) Ch. 8.1
- [33] Gotay M 1998 Obstructions to quantization, in: *The Juan Simo Memorial Volume* Eds. Marsden J. and Wiggins S. (New York: Springer) [math-ph/9809011](#)
 Gotay M 1998 On the Groenewold-Van Hove problem for R^{2n} [math-ph/9809015](#)
- [34] Kanatchikov I V 1998 From the DeDonder-Weyl Hamiltonian formalism to quantization of gravity in: *Current Topics in Mathematical Cosmology* (Proc. Int. Seminar, Potsdam, Germany, Mar 30-Apr 04 1998) eds. Rainer M and Schmidt H-J (Singapore: World Scientific) p. 457-67 [gr-qc/9810076](#)

- [35] Hořava P 1991 On a covariant Hamilton-Jacobi framework for the Einstein–Maxwell theory *Class. Quantum Grav.* **8** 2069-84
Also see: Krupka D and Štěpánková O 1982 On the Hamilton form in second order calculus of variations in: Proc. Meeting “Geometry and Physics” (Florence, Italy, Oct 1982) p. 85-101
- [36] Misner C W 1957 Feynman quantization of general relativity *Rev Mod Phys* **29** 497
- [37] Gotay M J 1991a A multisymplectic framework for classical field theory and the calculus of variations I. Covariant Hamiltonian formalism in: *Mechanics, Analysis and Geometry: 200 Years after Lagrange* ed. Francaviglia M (Amsterdam: North Holland) p. 203-35
Gotay M J 1991b A multisymplectic framework for classical field theory and the calculus of variations II. Space + time decomposition *Diff. Geom. and its Appl.* **1** 375-90
Śniatycki J 1984 The Cauchy data space formulation of classical field theory *Rep. Math. Phys.* **19** 407-22
- [38] Hatfield B 1992 *Quantum Field Theory of Point Particles and Strings* (Reading, MA: Addison-Wesley)
- [39] Castro C 1998 p-Brane quantum mechanical wave equations **hep-th/9812189**
- [40] Klauder J and Aslaksen E W 1970 Elementary model for quantum gravity *Phys. Rev.* **D2** 272-7
Also see: Klauder J 1969 Soluble models for quantum gravitation in: *Relativity* (Proc. Conf. Midwest 1969) ed. Carmeli M et al (New York: Plenum Press, New York) p. 1-17
Klauder J 1980 Path integrals for affine variables in: *Functional Integration* eds. Antoine J-P and Triapequi E (New York: Plenum Press) p. 101-19